Multi-District School Choice: Playing on Several Fields*

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Abstract

We extend the seminal model of Pathak and Sönmez (2008) to a setting with multiple school districts, each running its own separate centralized match, and focus on the case of two districts. In our setting, in addition to each student being either sincere or sophisticated, she is also either *constrained*—able to apply only to schools within her own district of residence—or *unconstrained*—able to choose any single district within which to apply. We show that several key results from Pathak and Sönmez (2008) qualitatively flip: A sophisticated student may prefer for a sincere student to become sophisticated, and a sophisticated student may prefer for her own district to use Deferred Acceptance over the Boston Mechanism, irrespective of the mechanism used by the other district. We furthermore investigate the preferences of students over the constraint levels of other students. Many of these phenomena appear abundantly in large random markets.

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1 Introduction

The Boston Mechanism (henceforth BM; also sometimes referred to as the "Immediate Acceptance" mechanism) is a widely used school-choice mechanism, especially in school-choice systems that were never (re)designed by economists or computer scientists. This mechanism first maximizes the number of applicants who get their first-choice school (breaking ties based on the priorities that students have at the different schools, i.e., based on the schools' "preferences"); then, subject to that, maximizes the number of applicants who get their second-choice school; then, subject to that, maximizes the number of applicants who get their third-choice school; and so forth. Despite being a very natural mechanism, BM suffers from various unattractive qualities, such as not being strategyproof and resulting in unstable matchings. Due to these and other shortcomings, there has been a push since the turn of the millenium (Abdulkadiroğlu et al., 2005b,a; Pathak and Sönmez, 2008) to replace BM with the better-behaved Deferred Acceptance mechanism (Gale and Shapley, 1962, henceforth DA) in school-choice systems.

One of the most compelling arguments given in favor of replacing BM with DA is the equity argument that originates in the seminal paper of Pathak and Sönmez (2008), which considers a setting with some students being sincere (i.e., uninformed and unstrategic, always reporting their true preferences) and some being sophisticated (i.e., informed and strategic, together playing a Nash equilibrium). That paper proves that sophisticated students weakly prefer BM over DA (which could be seen as a baseline that treats sincere and sophisticated students equally, due to its strategyproofness). This leads Pathak and Sönmez (2008) to view BM as weakly (and many times strictly) conferring an advantage to sophisticated students over sincere ones. Pathak and Sönmez (2008) furthermore prove that when BM is used, sophisticated students weakly prefer for sincere students to remain sincere, giving a plausible explanation as to why informed parent groups might not be likely to share their know-how with parents outside their groups or social circles.

A school district running a centralized matching mechanism is not an isolated capsule. Many districts, each running an independent centralized match, might exist next to each other, and some students might be able to effectively choose which district's match to participate in. For example, a 2005 report for the Berkeley Unified School District in California (Fried, 2005) estimated that between 7.8% and 12% of the district's high schoolers were "attending [the district] unofficially", and actually lived out-of-district. Choosing one's school district can thus be done without official permission (as in the case above) at personal risk,[1]

[1]See, for instance, Martin (2011) for more context on this illegal phenomenon, known as "boundary hopping" or "residency fraud," which at times has led to a prison sentence for parents.

or legally by moving to that district, an option many times available only to populations with greater financial resources.[2]

Neighbouring school districts are often independent of each other and might use different mechanisms. Due to the ability of some students to choose their school districts, one district switching its mechanism has the potential to change the multi-district equilibrium, changing students' strategies not only in terms of how they rank schools within a school district but also in terms of their choice of school district. In this paper, we examine the two predictions of Pathak and Sönmez (2008) that we describe above in a multi-district setting in which district choice (by the students who, for instance, possess the resources to officially relocate or are willing to risk punishment) is endogenized as part of the equilibrium. That is, in our setting, in addition to each student being either sincere or sophisticated, she is also either *constrained*—able to apply only to schools within her own district of residence—or *unconstrained*—able to strategically choose any single district within which to apply.[3] We prove that even when considering only two school districts, both of the predictions of Pathak and Sönmez (2008) that we describe above flip. That is, a sophisticated student may strictly prefer for her district to use DA over BM, irrespective of whether she is constrained or unconstrained and of the mechanism used by the other district. Furthermore, a sophisticated student may strictly prefer for some sincere student to become sophisticated. The latter phenomenon also appears abundantly in large random markets, that is, as the size of the market grows, a constant fraction of sophisticated students strictly prefer that at least some sincere students become sophisticated. Finally, we complete our investigation by asking whether some students might prefer for others to change their constraint type, e.g., whether an unconstrained student might prefer for another student to become unconstrained, or whether a constrained student might prefer for another student who resides in a different district to become constrained. We prove a strong "anything goes" result showing that every possible such combination is abundant in a large random market.

Our results are not without limitations. For one, consider the phenomena of sophisticated students strictly preferring DA over BM. While we show this phenomenon to be possible, it might be rare in random markets,[4] which could still lend credence to an argument in favor of DA over BM. Importantly, though, this argument becomes a quantitative issue of relative frequency rather than a qualitative issue of existence. Pathak and Sönmez (2008), in

[2]Comparing the prices of houses located near school district boundaries, Black (1999) estimates that parents are willing to pay 2.1% more to enroll their child in a district with a 5% higher mean test score, and Bayer et al. (2007) similarly estimate a 1.8% higher willingness to pay for homes in a district with an average test score that is higher by one standard deviation.

[3]In either case, if she is sophisticated, she can strategically order her submitted preference list over the schools in the district to which she applies.

[4]We do show that its frequency at the very least does not diminish as the market grows.

their study of the one-district setting, speculate that school-choice discussion organizations like the West Zone Parents Group may exist to coordinate behavior among sophisticated students without informing sincere ones. While our results indicate that in the multi-district setting it is feasible for such organizations to have incentives to inform some sincere students and increase their sophistication, these groups do not necessarily have a means of targeting specifically the sincere students who improve some sophisticated students' outcomes, again rendering the story less clear cut.

Our paper, therefore, is not intended to advocate for the use of BM. Rather, first and foremost, it serves to introduce a formal model of multi-district school choice and highlight that taking into account the broader landscape beyond only a single district may qualitatively change the analysis, including the arguments for or against the use of various mechanisms. Specifically, our results provide a proof-of-concept that highlights that district choice, which manifests as the real-life behavior of sending one's child to school in a desirable district by either using a false address or paying a premium to move, plays an important role in shaping the playing field for both sophisticated and sincere students. Our results should be interpreted as "anything goes" results, highlighting that when designing the specifics of a mechanism or market (even when choosing, for instance, between different ways to break ties in students' priorities at schools if these priorities are not strict), there are no one-size-fits-all solutions. Instead, one must weigh the specifics of the market in question, perhaps even more broadly defined than usually considered.

1.1 Related Work

The application of mechanism design to school choice originated in Abdulkadiroğlu and Sönmez (2003). Strategic opportunities in BM had been observed when this mechanism was first described in the economic literature (Abdulkadiroğlu et al., 2005b), and were subsequently shown in the lab (Chen and Sönmez, 2006) and in the field (Calsamiglia and Güell, 2018). Welfare arguments in favor of DA over BM have appeared in Ergin and Sönmez (2006) and Kojima (2008), culminating in the equity and fairness arguments of Pathak and Sönmez (2008). Several papers examine some of the predictions of Pathak and Sönmez (2008) in various extended models (still within a single district), such as with coarse priority structures (Abdulkadiroğlu et al., 2011; Babaioff et al., 2019) or with a finer classification of sophistication types (Zhang, 2021). Our large-market analysis methods are technically most closely related to those of Babaioff et al. (2019).

To our knowledge, ours is the first theoretical analysis of multi-district school choice. Closest to our work are previous papers that analyze different types of schools (such as charter, magnet, and private schools) coexisting with public schools. For instance, considering schools that are not district-run and can therefore choose to use their own admissions systems, Ekmekci and Yenmez (2019) analyze school incentives for participating in a unified enrollment system in a single district. Other papers assume there is no option for a unified enrollment system, and instead analyze a "slightly decentralized" mechanism that can be used to rematch students with the vacant seats that arise from schools of several types accepting the same student (Manjunath and Turhan, 2016; Turhan, 2019; Afacan et al., 2022). Akbarpour et al. (2022) study how exogenously varying the value of outside options affects behavior and preference over mechanisms. Although these threads of research bear some resemblance to the multi-district school choice problem that we study, there are a number of key differences. Ekmekci and Yenmez (2019) fix the students in the district and focus on the choice by schools of whether to participate in a unified enrollment system; we instead fix the schools in each of multiple districts and allow some students to choose in which district to apply. Meanwhile, Manjunath and Turhan (2016), Turhan (2019), and Afacan et al. (2022) allow every student to report rankings for each school type and choose between matches they receive; Akbarpour et al. (2022) similarly do not require students to forego their outside option in order to participate in a match. By contrast, we distinguish between students who can and cannot utilize district choice, and even those who can are only able to rank schools in their single chosen district (and therefore only receive one match, and cannot keep a guaranteed outside option). Our paper centers on how students endogenously choose their district of enrollment and thus affect the landscape of a school choice problem, a modeling decision that distinguishes this paper from the above prior work.

Finally, our investigation into the interplay between the choice of mechanism for one district and the multi-district equilibrium can be seen as contributing to a recent line of work on "partial mechanism design" (e.g., Philippon and Skreta, 2012; Tirole, 2012; Kang, 2023; see Kang and Muir, 2023, for a review).

2 Model

2.1 Standard Concepts

Employing much of the notation of Pathak and Sönmez (2008), we use the following standard concepts from the school choice literature.

2.1.1 Single-District School Choice

In a (single-district) *school choice problem*, there is a set of *students* $I = \{i_1, ..., i_n\}$ and a set of *schools* $S = \{s_1, ..., s_m\}$. Each student i has a strict *preference ordering* P_i over some subset of S, and i prefers remaining unassigned over being assigned to schools that are not in this subset. Each school s has a *capacity* of q_s seats, which is the maximum number of students that s can accept, and a strict *priority ordering* π_s over all students. The schools' priorities for students are *responsive* in the sense that:

- a school cannot reject students if it is not at capacity, and
- a school cannot accept a lower priority student over a higher priority student, regardless of which other students may or may not be accepted.

School priority orderings and capacities are public (e.g., set by policy). So that school assignments can be determined, each student submits a *rank-order list* (henceforth, *ROL*) of any number of schools, which may or may not match her actual preference ordering.

2.1.2 Mechanisms

A school choice mechanism uses students' submitted ROLs and schools' priority orderings and capacities to determine school assignments in a single district. Two such mechanisms are the *Boston Mechanism* (abbreviated as *BM*) and Deferred Acceptance (abbreviated as *DA*).

Definition 2.1. The Boston Mechanism (BM) (Abdulkadiroğlu et al., 2005b) operates in several rounds as follows:

- Round 1: Each student who submitted a non-empty ROL applies to the school she ranked 1st on her ROL. For each school, if there are at least as many seats available as applicants, the school (permanently) accepts every applicant. Otherwise, each school allocates seats to applicants based on the school's priority ordering up to its capacity, and rejects the remaining students for whom no seats remain.
- Round $k > 1$: Consider only students who have not yet been accepted to a school (i.e., the students who have been rejected by the schools 1st through $(k-1)$th on their ROLs). Each student who submitted an ROL of at least length k applies to the school she ranked kth. For each school, applicants are accepted or rejected in the same way as in Round 1, where the seats available are those that were not already filled in previous rounds. If a school has no seats available at the beginning of the round, it rejects all new applicants.

- This process terminates when every student has been either assigned a seat at a school or rejected by every school on her ROL, in which case she remains unassigned.

Observe that a student is immediately permanently accepted or rejected when she applies to a school in BM. For this reason, BM is also known as the Immediate Acceptance mechanism.

Definition 2.2. The Deferred Acceptance mechanism (Gale and Shapley, 1962) also operates in several rounds, but with only tentative acceptances until the very end, as follows:

- Round 1: Each student who submitted a non-empty ROL applies to the school ranked 1st on her ROL. For each school, if there are at least as many seats available as applicants, the school tentatively accepts every applicant. Otherwise, each school tentatively allocates seats to applicants based on the school's priority ordering up to its capacity, and (permanently) rejects the remaining students for whom no seats remain.

- Round $k > 1$: Consider only students who are not currently tentatively accepted at a school (i.e., the students who were rejected by a school in round $k-1$). Each of these students applies to the school highest on her ROL that has not already rejected her. For each school, new applicants are considered alongside tentatively accepted students. All of these students are compared based on the school's priority ordering and are tentatively accepted or permanently rejected in the same way as in Round 1, where all seats at the school are initially considered available.[5]

- This process terminates when every student has been either assigned a tentative seat at a school or rejected by every school on her ROL, in which case she remains unassigned. At this point, all tentative acceptances become permanent.

We say that a mechanism is *strategyproof* if truthful reporting is a dominant strategy for every student. BM is not strategyproof: a student may benefit from reporting an ROL that differs from her true preference ordering. DA is strategyproof (Dubins and Freedman, 1981; Roth, 1982): regardless of other students' reported ROLs, it is every student's dominant strategy to report her true preference ordering as her ROL.

2.2 Multi-District School Choice

In this paper, we extend the traditional school choice problem (henceforth, the *single-district school choice problem*) by considering multiple districts. Specifically, in a *multi-district school*

[5] Students who were tentatively accepted by the school in round $k-1$ are not conferred any advantage, and may still be permanently rejected by that school in round k.

choice problem, there is a set of *districts* numbered $\{1, ..., \ell\}$. Each student i resides in some district $d(i)$ and each school s is located in some district $d(s)$. A student's preference ordering may be over schools in multiple districts (including districts in which the student does not reside), and a school's priority ordering is over all students across all districts. A student's ROL may only contain schools from a single district, however; we call this the district in which she *enrolls*. Intuitively, this models the real-world setting wherein a student can only enroll in a single school district in a given year.

Each district uses its own school choice mechanism to determine school assignments of the students who enroll in the district to the schools that are located in the district. Different districts may use the same mechanism or different mechanisms; regardless, the school assignments for each district are combined to form the school assignments for the multi-district school choice problem as a whole.

2.3 Student Sophistication Types and Constraints Types

As in Pathak and Sönmez (2008), a student is either *sincere* or *sophisticated*; this is known as her *sophistication type*. Once a sincere student i determines she will enroll in some district j, she submits her preference ordering limited to schools in j (i.e., P_i with any schools not in j removed) as her ROL. In other words, a sincere student reports her true preferences over schools in the district she enrolls in. On the other hand, a sophisticated student can strategize by submitting any ROL over schools in the district she enrolls in. In addition to having a sophistication type, in the multi-district setting, a student is also either *constrained* or *unconstrained*; we refer to this as her *constraint type*. A constrained student i can only enroll in the district in which she resides, $d(i)$, while an unconstrained student can enroll in any (single) district.

Combining these two attributes, we have four categories of students: *sincere-constrained*, *sincere-unconstrained*, *sophisticated-constrained*, and *sophisticated-unconstrained*. The behavior of each of these is largely intuitive, with a sincere-constrained student reporting her true preferences over schools in her district of residence; a sophisticated-constrained student strategically choosing an ROL over schools in her district of residence; and a sophisticated-unconstrained student strategically choosing both a district to enroll in and an ROL to submit over schools in that district.

One might consider two different definitions for sincere-unconstrained students. A sincere-unconstrained student could enroll in the district of her first-choice school and then report her true preferences over schools in that district; such a student does not strategize at all. Alternatively, a sincere-unconstrained student could also know that she will report her true preferences over schools in whichever district she enrolls in, but strategically choose which district to enroll in. By the first definition, a sincere-unconstrained student who can never get assigned to her first-choice school will still enroll in the district of that school; in contrast, using the second definition, a sincere-unconstrained student would instead enroll in another district if she would get a preferable school assignment by reporting her true preferences over schools in that district. We believe that both of these definitions are have merit in different contexts, so we ensure that our results hold regardless of which definition is used. Where relevant, we demonstrate in our examples and theorems that the outcome is the same whether the first or second definition of sincere-unconstrained students is used.

2.4 Uniform $(n; k)$ model

Throughout this paper, we use examples of specific multi-district school choice problems to demonstrate particular phenomena, some of which stand in contrast to the propositions in Pathak and Sönmez (2008) that hold for the single-district setting. To analyze how frequently such phenomena occur, we consider large random two-district school choice problems inspired by the uniform models of Babaioff et al. (2019). In the *uniform* $(n; k)$ *model*, there are 2 districts labelled L and R.[6] Collectively, L and R contain n students $I = \{i_1, ..., i_n\}$ and n schools $S = \{s_1, s_2, ..., s_n\}$, each with unit capacity (i.e., $q_s = 1$ for all $s \in S$).

Each student is either sincere or sophisticated; is either constrained or unconstrained; and resides in either district L or district R. There are thus eight categories of students: One for each possible *sophistication type - constraint type - district of residence* combination. A student's category is drawn independently of all other students' categories, and there is a positive probability of a student's category being any of the eight possibilities. As such, there exists some $p > 0$ such that for each category, the probability of an arbitrary student being in this category is at least p. Each student's preference ordering over schools (which may include schools in any district) is drawn uniformly at random from among all (strict) possible preference orderings of length k.[7] Each student's preference ordering is independent of all other students' preference orderings, and of all students' categories.

Each school independently has probability $1/2$ of being located in district L and $1/2$ of being located in district R.[8] Finally, each school has a complete (strict) priority ordering over all students, drawn uniformly at random from the set of all such possible orderings, and independently of everything else. Thus, for any school and any two students i_a and i_b, the

[6] Later, we denote schools in L as $\ell_1, \ell_2, ...$ and schools in R as $r_1, r_2, ...$ for ease of reading. We use L and R rather than numbering the districts 1 and 2 for this reason.

[7] This is a special case of the procedure used to draw preference lists in Immorlica and Mahdian (2005).

[8] These probabilities need only be constant and nonzero values that sum to 1 for our results to hold, but we set them equal to avoid clutter.

probability that i_a has priority over i_b at that school is $1/2$.

3 Sophistication Types

In this section, we show that it is possible for a sophisticated student to prefer that (i.e., strictly benefit if) some sincere student becomes sophisticated. In fact, we prove that such students are abundant in large random markets. This result stands in contrast to Pathak and Sönmez (2008), which shows that in the single-district setting, all sophisticated students weakly suffer if any sincere student becomes sophisticated.

3.1 Example: A sophisticated student may prefer for a sincere student to become sophisticated

We first provide an illustrative example. Suppose that there are two districts with schools $\ell_1, \ell_2 \in L$ and school $r_1, r_2, r_3, r_4 \in R$, where each school has unit capacity. District L uses DA, while district R uses BM. Further suppose that there are five students i_1, i_2, i_3, i_4, and i_5.

The students' preference orderings, districts of residence, constraint types, and sophistication types are as follows.[9] Students whose preference orderings contain schools in only one district reside in that district and have arbitrary constraint types. Other students (those whose preference orderings contains schools in both districts) are unconstrained and reside in an arbitrary district.

$$\begin{aligned}
\text{(sincere) } i_1 :& \ell_1 \\
\text{(sincere) } i_2 :& \ell_1 \succ r_2 \succ r_1 \succ \ell_2 \\
\text{(sophisticated) } i_3 :& \ell_2 \succ r_3 \\
\text{(sincere) } i_4 :& r_2 \\
\text{(sincere) } i_5 :& r_1 \succ r_4
\end{aligned}$$

[9]The notation $s_a \succ s_b$ indicates a preference for school s_a over school s_b.

The schools' priority orderings include the following.[10]

$$\ell_1 : i_1 - i_2$$
$$\ell_2 : i_2 - i_3$$
$$r_1 : i_2 - i_5$$
$$r_2 : i_4 - i_2$$
$$r_3 : i_3$$
$$r_4 : i_5$$

We will show that i_3 prefers for i_2 to become sophisticated, and that it is irrelevant which definition of sincere-unconstrained is used.

First, consider the original setting where i_2 is sincere. Observe that regardless of which definition of sincere-unconstrained is used, i_2 will enroll in district L. If i_2 does not strategize at all, then i_2 will enroll in district L because her first choice school is in district L. If i_2 strategizes over districts but always reports a truthful ROL, then i_2 will see that enrolling in district R results in her being unassigned. This is because i_2 would not be matched in the first round of BM, during which both r_1 and r_2 would be filled. Because L uses DA, on the other hand, i_2 would not be matched in the first round but would still get matched with a school in her preference ordering. Therefore, i_2 enrolls in district L.

Since i_2 enrolls in district L, sophisticated-unconstrained student i_3 enrolls in district R (and ranks only r_3) to avoid being unmatched. The matching process results in i_1 assigned to ℓ_1, i_2 assigned to ℓ_2, i_3 assigned to r_3, i_4 assigned to r_2, and i_5 assigned to r_1. This is the unique Nash equilibrium outcome. Note that in this outcome, i_2 is assigned to i_2's fourth-choice school and i_3 is assigned to i_3's second-choice school.

Suppose instead that i_2 becomes sophisticated. Student i_2 has no chance of being assigned to her first- or second-choice schools, and is guaranteed admittance to r_1 if she enrolls in district R and ranks r_1 first, so she does so. Note that if i_2 does not rank r_1 first, then i_2 would not be admitted to r_1. Student i_3 therefore chooses to enroll in district L (and ranks only l_2), as this guarantees her admittance at ℓ_2 (and she would not be admitted there otherwise). The matching process results in i_1 assigned to ℓ_1, i_2 assigned to r_1, i_3 assigned to ℓ_2, i_4 assigned to r_2, and i_5 is assigned to r_4. This is the unique Nash equilibrium outcome. In this outcome, i_2 is assigned to i_2's third-choice school and i_3 is assigned to i_3's first-choice

[10] The notation $i_a - i_b$ indicates priority for student i_a over student i_b. Technically, a school's priority ordering must include all students. Here, for each school, we list only the priority ordering over students who find the school acceptable, as no other student would ever apply to the school. Other students could be placed anywhere in each school's priority ordering without affecting our results.

school. Therefore, i_3 prefers for i_2 to become sophisticated.

3.2 Large-Market Analysis

We generalize the example from Section 3.1 to the uniform $(n; 4)$ model. (The same analysis also works in the uniform $(n; k)$ model for any constant $k \geq 4$.) We say that a sophisticated student i_a prefers for a sincere student i_b to become sophisticated if i_a strictly prefers her match in every Nash equilibrium of the multi-district choice problem when i_b is sophisticated to her match in every Nash equilibrium of the multi-district choice problem when i_b is sincere. We show that there can be many sophisticated students who prefer for distinct sincere students to become sophisticated.

Theorem 3.1. *For every $p \in (0, 1)$, there exists $\tau > 0$ such that for any large enough n, in the uniform $(n; 4)$ model with one district using DA and the other using BM, there exists a set of sophisticated students of expected size at least τn where each sophisticated student strictly prefers for a distinct sincere student to become sophisticated, and weakly prefers for all other sincere students to become sophisticated.*

Proof. WLOG, let L be the district that uses DA and let R be the one that uses BM. We start by lower bounding the expected number of ordered quintuples of students $(i_1, i_2, i_3, i_4, i_5)$ that satisfy the following conditions (as in the example from Section 3.1):

1. **Conditions on sophistication types, constraint types, and districts of residence:**
   (a) i_1 is sincere and resides in L.
   (b) i_2 is sincere and unconstrained.
   (c) i_3 is sophisticated and unconstrained.
   (d) Both i_4 and i_5 are sincere and reside in R.
2. **Conditions on student preferences and school locations:**[11]
   (a) i_1 most prefers a school $\ell_1 \in L$.
   (b) i_5 most prefers a school $r_1 \in R$ and second most prefers a school $r_4 \in R$.
   (c) i_4 most prefers a school $r_2 \in R$.
   (d) i_3 most prefers a school $\ell_2 \in L$ and second most prefers a school $r_3 \in R$.

[11]These have been reordered from the example in Section 3.1 to match their order in the analysis below.

(e) i_2 has preference ordering $\ell_1 \succ r_2 \succ r_1 \succ \ell_2$.

(f) No other student finds $\ell_1, \ell_2, r_1, r_2, r_3$, or r_4 acceptable.

3. **Conditions on school priorities:**

(a) i_1 has priority over i_2 at ℓ_1.

(b) i_2 has priority over i_3 at ℓ_2.

(c) i_2 has priority over i_5 at r_1.

(d) i_4 has priority over i_2 at r_2.

As in the example in Section 3.1, in this situation, i_3 would strictly benefit from i_2 becoming sophisticated.

We want to determine the probability that this set of conditions occurs for a specific $(i_1, i_2, i_3, i_4, i_5)$. Each of the three sets of conditions is independent. The conditions on student sophistication types, constraint types, and districts of residence are satisfied with probability at least p^5. The conditions on student preferences and school locations are satisfied with probability

$$\left(\frac{1}{2}\right) \cdot \left(\frac{n-1}{n} \cdot \frac{1}{2} \cdot \frac{n-2}{n-1} \cdot \frac{1}{2}\right) \cdot \left(\frac{n-3}{n} \cdot \frac{1}{2}\right) \cdot \left(\frac{n-4}{n} \cdot \frac{1}{2} \cdot \frac{n-5}{n-1} \cdot \frac{1}{2}\right) \cdot \\ \cdot \left(\frac{1}{n(n-1)(n-2)(n-3)}\right) \cdot \left(\frac{n-6}{n} \cdot \frac{n-7}{n-1} \cdot \frac{n-8}{n-2} \cdot \frac{n-9}{n-3}\right)^{n-5}.$$

In the expression above, each of the six parenthetical expressions corresponds to one of the six conditions on student preferences and school locations, and represents the condition's probability, conditioned on the previous conditions. For sufficiently large n,[12] this probability can be lower bounded by[13]

$$\frac{1}{2^6} \cdot \frac{(n-4)!}{n!} \cdot \frac{1}{e^{24}} \cdot \frac{1}{2}.$$

Finally, the conditions on school priorities are satisfied with probability $1/2^4$.

The number of ordered quintuples of distinct students is $\frac{n!}{(n-5)!}$. List all such quintuples and let the jth element of the list be I_j. Let $\mathbb{1}_{I_j}$ be the indicator random variable that represents whether the students in I_j satisfy all of the above conditions. Then the expected

[12]Note that $\lim_{n\to\infty}(1 - \frac{r}{n-a})^{(n-b)}$, where a, b, and r are constants, is e^{-r}.

[13]The final $1/2$ is to accommodate for all the multiplicands of the form $\frac{n-a}{n-b}$ or $(1 - \frac{r}{n-a})^{(n-b)}$, which have a nonzero limit, not fully converging for finite large n.

number of quintuples of students that satisfy all conditions is

$$\mathbb{E}\left[\sum_{j=1}^{\frac{n!}{(n-5)!}} \mathbb{1}_{I_j}\right] \geq \frac{n!}{(n-5)!} \cdot p^5 \cdot \frac{1}{2^4} \cdot \frac{1}{2^7 \cdot e^{24}} \cdot \frac{(n-4)!}{n!} = \frac{p^5(n-4)}{2^{11} \cdot e^{24}}.$$

To conclude the proof, we observe that by construction, no student can participate in two different quintuples that satisfy the above conditions. Therefore, all such quintuples of students involve distinct students, so there are at least an expected $\frac{p^5}{2^{11} \cdot e^{24}} \cdot (n-4)$ sophisticated students who each strictly prefer for a distinct sincere student to become sophisticated. Also by construction, each such sophisticated student is unaffected by sincere students outside of her quintuple, and the sophistication types of students i_1, i_4, and i_5 in the quintuple does not affect the dynamic between students i_3 and i_2.[14] Thus, these sophisticated students also weakly prefer for all other sincere students to become sophisticated.

We choose[15] $\tau = \frac{p^5}{2^{12} \cdot e^{24}}$ to satisfy the theorem statement. □

4 Mechanism Choice

In this section, we show that another key result of Pathak and Sönmez (2008) no longer holds true in the multi-district setting. Specifically, we show that in the multi-district setting, a sophisticated student may strictly prefer for a district to use DA instead of BM. This is true regardless of whether the sophisticated student is constrained or unconstrained. In particular, we will give an example where the sophisticated student only finds schools in one district acceptable and prefers for that district to use DA.

4.1 Example: A sophisticated student may strictly prefer Deferred Acceptance

Our example is as follows. Suppose there are two districts with schools $\ell_1, \ell_2 \in L$ and school $r_1 \in R$, where each school has unit capacity. Further suppose that there are three students i_1, i_2, and i_3.

The students' preference orderings, districts of residence, constraint types, and sophistication types are as follows. Students i_1 and i_2 reside in L and have arbitrary constraint

[14] This is because i_1 and i_4 will always be assigned to their first choice schools, and i_5 will either be assigned to r_1 if i_2 is sincere or r_4 if i_2 is sophisticated. r_4 is part of this example to preclude i_5 from affecting the other students in the quintuple if she were to become sophisticated.

[15] We made no attempt to optimize this value, as our goal was only to ascertain linearity in n of the number of such students.

types. Student i_3 is unconstrained and resides in an arbitrary district.

$$\text{(sincere) } i_1 : \ell_1 \succ \ell_2$$
$$\text{(sophisticated) } i_2 : \ell_2 \succ \ell_1$$
$$\text{(sophisticated) } i_3 : \ell_2 \succ r_1$$

The schools' priority orderings include the following:

$$\ell_1 : i_2 - i_1$$
$$\ell_2 : i_1 - i_3 - i_2$$
$$r_1 : i_3$$

Observe that because i_3 is the only student who finds any school in district R acceptable, the mechanism used by district R is irrelevant.

We will show that i_2 prefers district L to use DA instead of BM. First, assume that district L is using BM. As sincere student i_1 will not apply to ℓ_2 in the first round, sophisticated student i_3 will choose to apply to ℓ_2 in the first round, guaranteeing i_3's acceptance at ℓ_2. Sophisticated student i_2 will realize she has no chance at ℓ_2 and will instead apply to ℓ_1. This process results in i_1 unassigned, i_2 assigned to ℓ_1, and i_3 assigned to ℓ_2, which is the unique Nash equilibrium outcome. Note that in this Nash equilibrium outcome, i_2 is assigned to her second-choice school.

Suppose instead that district L uses DA. Assume first that student i_2 uses her dominant strategy of ranking ℓ_2 above ℓ_1. This induces student i_3 to enroll in R instead of L, as enrolling in L would result in i_3 being unassigned. Specifically, i_3 could "knock out" i_2 from ℓ_2, but in that case i_2 would knock out i_1 from ℓ_1 and i_1 would in turn knock out i_3 from ℓ_2. This constitutes a Nash equilibrium. Now consider the other possible strategies for i_2. If she ranks ℓ_1 first, then she is assigned to ℓ_1 for sure. This would lead to i_1 being matched to ℓ_2 and hence i_3 enrolls in R and is matched to r_1; in this case, i_2 has a profitable deviation to ranking ℓ_2 first, so this is not a Nash equilibrium. Finally, if i_2 ranks only ℓ_2, then it is easy to see that i_3 enrolls in L and is matched to ℓ_2, and i_2 becomes unassigned; in this case, i_2 has a profitable deviation of ranking ℓ_1 somewhere in her preferences, so this is also not a Nash equilibrium. Altogether, the unique Nash equilibrium outcome has i_1 assigned to ℓ_1, i_2 assigned to ℓ_2, and i_3 assigned to r_1. In this Nash equilibrium outcome, i_2 is assigned to her first-choice school, which is a strict improvement for her compared to when district L uses BM.

A key feature of this example is that there exists a cycle within the preferences of i_1

and i_2. This causes i_3 to get knocked out of ℓ_2 when i_3 applies to district L and district L is using DA. We show in Appendix A that this example can be generalized to include a cycle with additional sophisticated students, in which each sophisticated student within the cycle similarly strictly prefers for her district to use DA.

4.2 Large-Market Analysis

We generalize the example from Section 4.1 to the uniform $(n;2)$ setting. (The same analysis also works in the uniform $(n;k)$ model for any constant $k \geq 2$.) We say that a sophisticated student i prefers for a district d to use DA if i strictly prefers her match in every Nash equilibrium of the multi-district choice problem when d uses DA to her match in every Nash equilibrium of the multi-district choice problem when d uses BM. We show that there can be a constant number of sophisticated students who each prefer for the district that contains her entire preference list to use DA.

Theorem 4.1. *For every $p \in (0,1)$, there exists $\tau > 0$ such that for any large enough n, in the uniform $(n;2)$ model there exists a set of sophisticated students of expected size at least τ where each sophisticated student strictly prefers for the district that contains her entire preference list to use DA rather than BM, regardless of the mechanism used by the other district.*

Proof. Let the two districts be L and R. We start by lower bounding the expected numbered of ordered triplets of students (i_1, i_2, i_3) that satisfy the following conditions (as in the example from Section 4.1):

1. **Conditions on sophistication types, constraint types, and districts of residence:**
   (a) i_1 is sincere and resides in L.
   (b) i_2 is sophisticated and resides in L.
   (c) i_3 is sophisticated and unconstrained.
2. **Conditions on student preferences and school locations:**
   (a) i_1 most prefers a school $\ell_1 \in L$, and next prefers a school $\ell_2 \in L$.
   (b) i_2 has preference ordering $\ell_2 \succ \ell_1$.
   (c) i_3 most prefers ℓ_2 and next prefers a school $r_1 \in R$.
   (d) None of the other $n-3$ students finds ℓ_1, ℓ_2, or r_1 acceptable.

3. **Conditions on school priorities:**

   (a) i_2 has priority over i_1 at ℓ_1.

   (b) At ℓ_2, i_1 has priority over i_3, who in turn has priority over i_2.

As in the example in Section 4.1, in this situation, i_2 would strictly benefit from L using DA rather than BM.

We want to determine the probability that this set of conditions occurs for a specific (i_1, i_2, i_3). Each of the three sets of conditions is independent. The conditions on sophistication types, constraint types, and districts of residence are satisfied with probability at least p^3. The conditions on student preferences and school locations are satisfied with probability

$$\left(\frac{1}{2} \cdot \frac{1}{2}\right) \cdot \left(\frac{1}{n} \cdot \frac{1}{n-1}\right) \cdot \left(\frac{1}{n} \cdot \frac{n-2}{n-1} \cdot \frac{1}{2}\right) \cdot \left(\frac{n-3}{n} \cdot \frac{n-4}{n-1}\right)^{n-3}.$$

In the expression above, each of the four parenthetical expressions corresponds to one of the four conditions on student preferences and school locations, and represents the condition's probability, conditioned on the previous conditions. For sufficiently large n, the above expression can be lower bounded by

$$\frac{1}{2^3} \cdot \frac{1}{n(n-1)(n-2)} \cdot \frac{1}{e^6} \cdot \frac{1}{2}.$$

Finally, the conditions on school priorities are satisfied with probability $\frac{1}{2} \cdot \frac{1}{6} = \frac{1}{2^2 \cdot 3}$.

The number of ordered triplets of distinct students is $n(n-1)(n-2)$. List all such triplets and let the jth element of the list be I_j. Let $\mathbb{1}_{I_j}$ be the indicator random variable which represents whether the students in I_j satisfy all of the conditions. Then the expected number of triplets of students that satisfy all conditions is

$$\mathbb{E}\left[\sum_{j=1}^{n(n-1)(n-2)} \mathbb{1}_{I_j}\right] \geq n(n-1)(n-2) \cdot p^3 \cdot \frac{1}{2^2 \cdot 3} \cdot \frac{1}{2^4 \cdot e^6} \cdot \frac{1}{n(n-1)(n-2)} = \frac{p^3}{2^6 \cdot 3 \cdot e^6}.$$

To conclude the proof, we observe that by construction, no student can participate in two different triplets that satisfy the above conditions. Therefore, all such triplets of students involve distinct students, so there are at least an expected $\frac{p^3}{2^6 \cdot 3 \cdot e^6}$ sophisticated students who each strictly prefer for the district that contains their entire preference list to use DA over BM. We choose $\tau = \frac{p^3}{2^6 \cdot 3 \cdot e^6}$ to satisfy the theorem statement. □

5 Constraint Types

In the previous two sections, we have shown that two predictions made by Pathak and Sönmez (2008) no longer hold in multi-district settings. The first of these predictions revolved around whether a sophisticated student might prefer for another student to change their sophistication type in a particular way (specifically, becoming sophisticated). In our setting, students are not only characterized by their sophistication type but also by their constraint type, and hence it is natural to ask whether and when some students might prefer for some other students to change their constraint type.

We find a strong "anything goes" result here: For any combination of constraint types for two students, it might be the case that the first student strictly prefers for the constraint type of the second student to change, and this is furthermore abundant in large random markets. This holds regardless of the sophistication types of the two students, regardless of whether or not they reside in the same district, and regardless of the mechanisms used by the two districts. (We state this result in the uniform $(n; 2)$ model; however, similarly to our previous results, the same analysis also works in the uniform $(n; k)$ model for any constant $k \geq 2$.) We say that a student i_a prefers for student i_b to change her constraint type if i_a is strictly worse off in every Nash equilibrium of the multi-district school choice problem when i_b has her given constraint type compared to every Nash equilibrium of the multi-district choice problem when i_b has the opposite constraint type.

Theorem 5.1. *For every $p \in (0, 1)$, there exists $\tau > 0$ such that for every pair of sophistication types s_1 and s_2, every pair of constraint types c_1 and c_2, and for any large enough n, in the uniform $(n; 2)$ model with the districts using any combination of matching mechanisms, there exists a set of expected size at least τn of students of sophistication type s_1 and constraint type c_1 where each student in the set strictly prefers for a distinct student of sophistication type s_2 and constraint type c_2 from the same district to change her constraint type. Furthermore, this also holds if "from the same district" is replaced with "from another district."*

As it turns out, two types of constructions suffice to cover all of the various combinations of constraint types, sophistication types, districts of residence, and mechanisms in Theorem 5.1. Let i_1 and i_2 be two students, where i_1 is the student who prefers for i_2 to change her constraint type. Both constructions involve i_2 vacating her seat at a school s—either because she becomes constrained and s is not in her district of residence, or because she becomes unconstrained and would rather enroll in another district.

The simpler of the two constructions has i_1 filling the vacancy left by i_2 at s. This construction can be used in cases where i_1 is unconstrained, as well as in cases where i_1

is constrained and the school s is in i_1's district of residence. The second, slightly more elaborate construction covers the remaining cases, in which i_1 is constrained and the school s is outside i_1's district of residency. These cases, in which our results are arguably more surprising at first glance, include situations where i_2 resides in the same district as (the constrained) i_1 and vacates a spot in another district when i_2 becomes constrained, as well as situations where i_2 resides in a different district than i_1 and vacates a spot in that district when i_2 becomes unconstrained. In such cases, we introduce a third, unconstrained student, i_3, who takes the seat vacated by i_2 and therefore vacates a seat in the district of i_1, which i_1 in turn gets to fill.

We now prove two cases of Theorem 5.1—one using each of the two constructions. The proof of each of the other cases of Theorem 5.1 is completely analogous to the proof of one of these two cases, as sketched above. We start by demonstrating the first, simpler construction.

Lemma 5.2. *For every $p \in (0,1)$, there exists $\tau > 0$ such that for any large enough n, in the uniform $(n;2)$ model with the districts using any combination of matching mechanisms, there exists a set of unconstrained students of expected size at least τn where each unconstrained student strictly prefers for a distinct constrained student to become unconstrained.*

Proof. Let the two districts be L and R. We start by lower bounding the expected number of ordered pairs of students (i_1, i_2) that satisfy the following conditions:

1. i_1 is unconstrained, and i_2 is constrained and resides in district L.
2. i_1 most prefers a school $\ell \in L$.
3. i_2 most prefers a school $r \in R$, and next prefers ℓ.
4. No other student finds ℓ or r acceptable.
5. i_2 is higher in the priority order than i_1 at ℓ.

Under these conditions, i_1 would strictly benefit from i_2 becoming unconstrained, as i_2 would switch from enrolling in district L to enrolling in district R, which would open up school ℓ for i_1.

We want to determine the probability that this set of conditions occurs for a specific i_1 and i_2. Conditioned on all previous conditions, the first condition occurs with probability at least p^2, the second condition occurs with probability at least $\frac{1}{2}$, and the third condition occurs with probability $\frac{n-1}{n} \cdot \frac{1}{2} \cdot \frac{1}{n-1}$. The fourth condition occurs with probability

$\left(\frac{n-2}{n} \cdot \frac{n-3}{n-1}\right)^{n-2}$. Finally, the fifth condition occurs with probability $\frac{1}{2}$. For sufficiently large n, this probability can be lower bounded by

$$\frac{p^2}{2^3} \cdot \frac{1}{n} \cdot \left(\frac{n-2}{n} \cdot \frac{n-3}{n-1}\right)^{n-2} \geq \frac{p^2}{2^4 e^4 (n-1)}.$$

The number of ordered pairs of students is $n(n-1)$. List all such pairs and let the jth element of the list be I_j. Let $\mathbb{1}_j$ be the indicator random variable which represents whether the students in I_j satisfy all of the above conditions. Then the expected number of pairs of students that satisfy all conditions is

$$\mathbb{E}\left[\sum_{j=1}^{n(n-1)} \mathbb{1}_j\right] \geq n(n-1) \cdot \frac{p^2}{2^4 e^4 (n-1)} = \frac{p^2}{2^4 e^4} \cdot n.$$

To conclude the proof, we observe that by construction, no student can participate in two different pairs that satisfy the above conditions. Therefore, all such pairs of students involve distinct students, so there are at least an expected $\frac{p^2}{2^4 e^4} \cdot n$ unconstrained students who each strictly prefer for a distinct constrained student to become unconstrained. We choose $\tau = \frac{p^2}{2^4 e^4}$ to satisfy the theorem statement. □

We now demonstrate the use of the second, slightly more elaborate, construction.

Lemma 5.3. *For every $p \in (0,1)$, there exists $\tau > 0$ such that for any large enough n, in the uniform $(n;2)$ model with the districts using any combination of matching mechanisms, there exists a set of constrained students of expected size at least τn where each constrained student strictly prefers for a distinct constrained student in another district to become unconstrained.*

Proof. Let the two districts be L and R. We start by lower bounding the expected number of ordered triplets of students (i_1, i_2, i_3) that satisfy the following conditions:

1. i_1 is constrained and resides in district L, i_2 is constrained and resides in district R, and i_3 is unconstrained.
2. i_1 most prefers a school $\ell_1 \in L$.
3. i_2 most prefers a school $\ell_2 \in L$, and next most prefers a school $r_1 \in R$.
4. i_3 most prefers r_1, and next most prefers ℓ_1.
5. No other student finds ℓ_1, ℓ_2, or r_1 acceptable.

6. i_3 is higher in the priority order than i_1 at ℓ_1, and i_2 is higher in the priority order than i_3 at r_1.

Under these conditions, i_1 would strictly benefit from i_2 becoming unconstrained, as i_2 would switch from enrolling in district R to enrolling in district L. As a result, i_3 would switch from enrolling in district L to enrolling in district R, which would open up school ℓ_1 for i_1.

We want to determine the probability that this set of conditions occurs for a specific (i_1, i_2, i_3). Conditioned on all previous conditions, the first condition occurs with probability at least p^3, the second condition occurs with probability at least $\frac{1}{2}$, the third condition occurs with probability $(\frac{n-1}{n} \cdot \frac{1}{2} \cdot \frac{n-2}{n-1} \cdot \frac{1}{2})$, and the fourth condition occurs with probability $(\frac{1}{n} \cdot \frac{1}{n-1})$. The fifth condition occurs with probability $\left(\frac{n-3}{n} \cdot \frac{n-4}{n-1}\right)^{n-3}$. Finally, the sixth condition occurs with probability $\frac{1}{2^2}$. For sufficiently large n, this probability can be lower bounded by

$$\frac{p^3}{2^5} \cdot \left(\frac{n-1}{n} \cdot \frac{n-2}{n-1}\right) \cdot \left(\frac{1}{n} \cdot \frac{1}{n-1}\right) \cdot \left(\frac{n-3}{n} \cdot \frac{n-4}{n-1}\right)^{n-3} \geq \frac{p^3}{2^6 e^6 (n-1)(n-2)}.$$

The number of ordered triplets of students is $n(n-1)(n-2)$. List all such triplets and let the jth element of the list be I_j. Let $\mathbb{1}_j$ be the indicator random variable that represents whether the students in I_j satisfy all of the above conditions. Then the expected number of triplets of students that satisfy all conditions is

$$\mathbb{E}\left[\sum_{j=1}^{n(n-1)(n-2)} \mathbb{1}_j\right] \geq n(n-1)(n-2) \cdot \frac{p^3}{2^6 e^6 (n-1)(n-2)} = \frac{p^3}{2^6 e^6} \cdot n.$$

To conclude the proof, we observe that by construction, no student can participate in two different triplets that satisfy the above conditions. Therefore, all such triplets of students involve distinct students, so there are at least an expected $\frac{p^3}{2^6 e^6} \cdot n$ constrained students who each strictly prefer for a distinct constrained student in another district to become unconstrained. We choose $\tau = \frac{p^3}{2^6 e^6}$ to satisfy the theorem statement. □

6 Discussion

In this paper, we show that several key results regarding sincere and sophisticated students from the seminal paper of Pathak and Sönmez (2008) no longer hold in a multi-district setting. This highlights the importance of weighing the specifics of the market in question when designing centralized mechanisms, where "the market in question" should be defined very broadly, perhaps more so than customarily considered.

Several aspects of our model and results would potentially benefit from further research. First, although the idea of district choice is motivated by students and their families boundary hopping or paying a premium to move, we do not explicitly model the associated (financial or risk-taking) costs. Future research could impose a price on choosing a district other than one's own district of residence, which would then factor into students' strategic considerations. For this to be effective, it would also be necessary to think about student satisfaction with different school assignments using cardinal utilities rather than ordinal preferences; modeling such costs is therefore outside the scope of this paper.

While Theorem 4.1 establishes an at least constant frequency of sophisticated students who strictly prefer DA over BM even in a large random market, it is our only theorem that does not prove the abundance of such students (i.e., an expected constant fraction of all sophisticated students in a large random market). Even if we extend the proof of Theorem 4.1 to also take into account all cycles of the form discussed in Appendix A, the same proof technique would yield only constant frequency. To achieve a linear, or even super-constant frequency, one would have to, for example, find a way for the existence of some cycle to be sufficient for a super-constant number of sophisticated students outside the cycle to strictly prefer BM over DA. While we conjecture that this is not possible, ruling this out seems to be related to the question of whether Deferred Acceptance "circuits" can efficiently encode computational circuits in which various wires split, a question that was resolved negatively by Cook et al. (2014). It is plausible that computational-complexity-theoretic tools such as those used in that paper might be leveraged to prove the asymptotic tightness of the bound in Theorem 4.1. We conjecture this bound to be tight (perhaps up to logarithmic factors if preference list lengths are not held constant), but leave the verification of this conjecture for future work.

References

A. Abdulkadiroğlu and T. Sönmez. School choice: A mechanism design approach. *American Economic Review*, 93(3):729–747, 2003.

A. Abdulkadiroğlu, P. A. Pathak, and A. E. Roth. The New York City high school match. *American Economic Review*, 95(2):364–367, 2005a.

A. Abdulkadiroğlu, P. A. Pathak, A. E. Roth, and T. Sönmez. The Boston public school match. *American Economic Review*, 95(2):368–371, 2005b.

A. Abdulkadiroğlu, Y.-K. Che, and Y. Yasuda. Resolving conflicting preferences in school choice: The "Boston mechanism" reconsidered. *American Economic Review*, 101(1):399–410, 2011.

M. O. Afacan, P. Evdokimov, R. Hakimov, and B. Turhan. Parallel markets in school choice. *Games and Economic Behavior*, 133:181–201, 2022.

M. Akbarpour, A. Kapor, C. Neilson, W. Van Dijk, and S. Zimmerman. Centralized school choice with unequal outside options. *Journal of Public Economics*, 210:104644, 2022.

M. Babaioff, Y. A. Gonczarowski, and A. Romm. Playing on a level field: Sincere and sophisticated players in the Boston mechanism with a coarse priority structure. In *Proceedings of the 20th ACM Conference on Economics and Computation (EC)*, page 345, 2019.

P. Bayer, F. Ferreira, and R. McMillan. A unified framework for measuring preferences for schools and neighborhoods. *Journal of Political Economy*, 115(4):588–638, 2007.

S. E. Black. Do better schools matter? parental valuation of elementary education. *The Quarterly Journal of Economics*, 114(2):577–599, 1999.

C. Calsamiglia and M. Güell. Priorities in school choice: The case of the Boston mechanism in Barcelona. *Journal of Public Economics*, 163:20–36, 2018.

Y. Chen and T. Sönmez. School choice: An experimental study. *Journal of Economic Theory*, 127(1):202–231, 2006.

S. A. Cook, Y. Filmus, and D. T. M. Le. The complexity of the comparator circuit value problem. *ACM Transactions on Computation Theory (TOCT)*, 6(4):1–44, 2014.

L. E. Dubins and D. A. Freedman. Machiavelli and the Gale-Shapley algorithm. *The American Mathematical Monthly*, 88(7):485–494, 1981.

M. Ekmekci and M. B. Yenmez. Common enrollment in school choice. *Theoretical Economics*, 14(4):1237–1270, 2019.

H. Ergin and T. Sönmez. Games of school choice under the Boston mechanism. *Journal of Public Economics*, 90(1-2):215–237, 2006.

R. Fried. Attending to the bottom line: Boosting district revenue and enhancing educational mission through interdistrict enrollment & attendance policy. An advanced policy analysis prepared for the Berkeley unified school district, Berkeley, California, 2005. URL `https://newspack-berkeleyside-cityside.s3.amazonaws.com/wp-content/uploads/2014/04/Attending-to-the-Bottomline-Final.pdf`.

D. Gale and L. Shapley. College admissions and the stability of marriage. *American Mathematical Monthly*, 69(1):9–15, 1962.

N. Immorlica and M. Mahdian. Marriage, honesty, and stability. In *Proceedings of the Sixteenth Annual ACM-SIAM Symposium on Discrete Algorithms*, SODA '05, page 53–62. Society for Industrial and Applied Mathematics, 2005.

Z. Kang and E. Muir. Partial mechanism design and incomplete-information industrial organization. Tutorial, The 24th ACM Conference on Economics and Computation (EC), 2023. URL `https://ziyangkang.com/ec'23-tutorial`.

Z. Y. Kang. The public option and optimal redistribution. Mimeo, 2023.

F. Kojima. Games of school choice under the Boston mechanism with general priority structures. *Social Choice and Welfare*, 31:357–365, 2008.

V. Manjunath and B. Turhan. Two school systems, one district: What to do when a unified admissions process is impossible. *Games and Economic Behavior*, 95:25–40, 2016.

M. Martin. Mother jailed for school fraud, flares controversy. Tell Me More, NPR, 2011. URL `https://www.npr.org/2011/01/28/133306180/Mother-Jailed-For-School-Fraud-Flares-Controversy`.

P. A. Pathak and T. Sönmez. Leveling the playing field: Sincere and sophisticated players in the Boston mechanism. *American Economic Review*, 98(4):1636–1652, 2008.

T. Philippon and V. Skreta. Optimal interventions in markets with adverse selection. *American Economic Review*, 102(1):1–28, 2012.

A. E. Roth. The economics of matching: Stability and incentives. *Mathematics of operations research*, 7(4):617–628, 1982.

J. Tirole. Overcoming adverse selection: How public intervention can restore market functioning. *American Economic Review*, 102(1):29–59, 2012.

B. Turhan. Welfare and incentives in partitioned school choice markets. *Games and Economic Behavior*, 113:199–208, 2019.

M. Yin. Multi-district school choice: When sincere students stay and sophisticated students stray. Underaduate thesis, Harvard University, 2022.

J. Zhang. Level-k reasoning in school choice. *Games and Economic Behavior*, 128:1–17, 2021.

A Cycles where Multiple Sophisticated Students Prefer Deferred Acceptance

In this appendix, we generalize the example in Section 4.1 to show that there may be multiple sophisticated students who all prefer that a district uses DA when their preferences form a cycle including exactly one sincere student.

Suppose again there are two districts L and R with schools $\ell_1, \ell_2, ..., \ell_x \in L$ and school $r_1 \in R$, where all schools have capacity 1. Further suppose that there are $x+1$ students $i_1, i_2, ..., i_{x+1}$. Student i_1 is sincere and constrained, while students $i_2, ..., i_{x+1}$ are all sophisticated and unconstrained.

The schools' priority orderings include the following:

$$
\begin{aligned}
\ell_1 :& i_2 - i_1 \\
\ell_2 :& i_3 - i_2 \\
&\vdots \\
\ell_{x-1} :& i_x - i_{x-1} \\
\ell_x :& i_1 - i_{x+1} - i_x \\
r_1 :& i_{x+1}
\end{aligned}
$$

The students' preferences are as follows:

$$
\begin{aligned}
i_1 :& \ell_1 \succ \ell_x \\
i_2 :& \ell_2 \succ \ell_1 \\
&\vdots \\
i_x :& \ell_x \succ \ell_{x-1} \\
i_{x+1} :& \ell_x \succ r_1
\end{aligned}
$$

Note that the preferences of students $i_1, ..., i_x$ are cyclical. Student i_{x+1} most prefers the second most preferred school of sincere student i_1, and second most prefers the only school in district R. Student i_{x+1} is also the only student who finds r_1 acceptable, which implies that the matching mechanism used by R is irrelevant.

We will show that students $i_2, ..., i_x$ all prefer district L to use DA instead of BM. First, assume that district L is using BM. As sincere student i_1 will not apply to ℓ_x in the first round, sophisticated student i_{x+1} will choose to apply to his first choice ℓ_x in the first round, guaranteeing acceptance at ℓ_x for himself. Sophisticated student i_x will realize she has no chance at ℓ_x and will instead apply to ℓ_{x-1}. This initiates a chain reaction in which all students $i_2, ..., i_x$ end up applying to their second choice school, as each would not be accepted by their first choice school. This process results in i_1 unassigned, $i_2, ..., i_x$ each assigned to their second choice schools $\ell_1, ..., i_{x-1}$ respectively, and i_{x+1} assigned to ℓ_x.

Now suppose instead that district L uses DA. Students $i_2, ..., i_x$ then each have a dominant strategy of ranking truthfully, as each will still be accepted to her second-choice school in a later round if rejected by her first choice school. This induces student i_{x+1} to enroll in R instead of L, as enrolling in L would result in i_{x+1} being unassigned. Note that if i_{x+1} chooses to enroll in L, i_{x+1} would end up being knocked out of ℓ_x by i_1, and would therefore be unassigned as i_{x+1} finds no other schools in L acceptable. Therefore, the unique Nash equilibrium has $i_1, ..., i_x$ each assigned to their first choice schools $\ell_1, ..., i_x$ respectively, and i_{x+1} assigned to r_1. In this Nash equilibrium, sophisticated students $i_2, ..., i_x$ are all assigned to their first choice schools, which is a strict improvement for each compared to when district L uses BM.